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AMERICAN SOCIETY OF CIVIL ENGINEERS.
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TRANSACTIONS.

No. 822.

THEORY OF THE IDEAL COLUMN.

By Wm. Cain, M. Am. Soc. C. E.

Presented December 1st, 1897.

WITH DISCUSSION.

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The columns to be examined will be regarded as "ideal" or as prismatic homogeneous columns, having the force P applied at one

end, in the direction of the axis or line through the centers of gravity of the cross-sections.

In Fig. 1, $A\ C\ B$ represents the original straight axis of the column, and the force P is supposed applied at A in the direction $A\ C\ B$. It is to be understood throughout that the force P is never to be so large as to cause the stress in any fiber to exceed the limit of elasticity.

First, let the column be supposed pivoted at the ends at A and B. As the column is straight and homogeneous, the force P will compress it, so that its primitive length, $A\ C\ B = a$ ins., is changed to $A'\ C\ B = a_1$ ins.; but the column will remain straight, as the stress is uniform on each cross-section. The weight of the column is neglected in this investigation. Calling $A =$ the area of cross-section in square inches, and $E =$ the modulus of elasticity in pounds per square inch, $A'\ C\ B = a_1 = a\left(1 - \frac{P}{E\,A}\right)$.

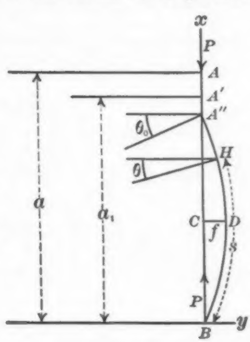

FIG. 1.

Now, suppose lateral forces to bend the column; if P is large enough, it remains bent after the lateral forces are removed, and the axis takes the position $A''\ H\ D\ B$ under the force P alone.

As the length of the axis is not altered by flexure, the length of the curved axis $A''\ H\ D\ B = a_1 = a\left(1 - \frac{P}{E\,A}\right)$.

In Fig. 1 call:

$I =$ moment of inertia of cross-section at H about an axis projected in H,

$\theta =$ angle (in circular measure) the section at H makes with its original horizontal direction,

$\theta_0 =$ value of θ at top of bent column A'',

$s =$ length of arc $B\ D\ H$,

$\rho =$ radius of curvature at $H = \frac{d\,s}{d\,\theta}$,

$f =$ maximum deflection $= C\ D$.

The origin of co-ordinates will be taken at B; x vertical (along primitive axis) and y horizontal.

If the primitive straight axis $A\ C\ B$ is divided into N equal parts, each equal to $\frac{a}{N}$, after compression of the axis from the length $A\ C\ B$ $= a$ to the length $A'\ C\ B = a_1$, each of these parts will now have the length $\frac{a_1}{N}$, and the same is true for each of the N equal parts into which the bent axis $A''\ H\ D\ B$ is divided, since the arc $A''\ H\ D\ B =$ $A'\ C\ B = a_1$.

In Fig. 2 is shown a portion of the bent column. The original length $H\ J = \frac{a}{N}$ of axis has been changed to length of arc $H\ H' = \frac{a_1}{N}$. On drawing a section $F\ F'$ through H' parallel to that at H, making $F\ H' = G\ H' = V =$ distance from axis to most compressed fiber, then on bending (after the uniform compression has been exerted that changes length $\frac{a}{N}$ of axis to $\frac{a_1}{N}$), the section $F\ F'$ at H', originally parallel to that at H, rotates, relatively to it, to position $G\ H'\ G'$, the points F and F' describing, relatively to G and G', circular arcs $F\ G$ and $F'\ G'$. Call length of arc $F\ G = K$.

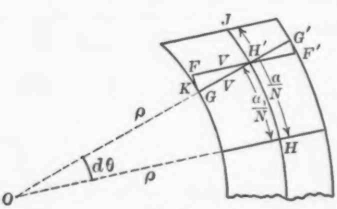

FIG. 2.

For a small deflection f (Fig. 1), the component of P parallel to the section at H is very small and may be neglected, so that the sections at H and H' (Fig. 2), originally normal to the axis, remain so after flexure. The angle $d\theta$ between the final sections at H and $H' =$ angle $F\ H'\ G$. If the arc $H\ H'$ is regarded as a circular arc, described with radius ρ from center O, which is admissible, as N will be supposed to increase indefinitely in going to the limit, it is seen that the circular sectors $O\ H\ H'$ and $F\ H'\ G$ are similar, having the same central angle; hence,

$$\rho : \frac{a_1}{N} : : V : K \therefore \rho = \frac{a_1\ V}{N\ K}.$$

It is seen from the figure that the bending alone does not change the length of any portion of the axis as $H\ H'$ (Fig. 2), as was assumed above.

If S designates the stress in the fiber at G of cross-section unity, due to bending alone, then $S = M\frac{V}{I}$ by a well-known formula, where

M = bending moment $= P\,y$, and y (Fig. 1) is the ordinate to the axis at the section considered.

By another standard formula,

K = change of length in fiber at G (Fig. 2).

$= S \times$ old length of fiber $HJ \div E$.

$$= \frac{S\frac{a}{N}}{E} = M\frac{V}{I}\,\frac{a}{N\,E}$$

$$\therefore\ \rho = \frac{a_1}{a}\,\frac{E\,I}{M} = \frac{d\,s}{d\,\theta}$$

The fact that, by the usual hypothesis, the change of length K, due to flexure only, varies with the stress and the old length of fiber has been especially emphasized by Henry S. Prichard, M. Am. Soc. C. E., in his notable contribution to the theory of "The Ideal Column."* The author's indebtedness to Mr. Prichard, in the previous analysis, will be evident on comparison.

In the correct value of $\rho = \frac{d\,s}{d\,\theta}$ given above, it is seen that N has been eliminated, hence the result is true at the limit, as N indefinitely increases and H' approaches H indefinitely.

From the value of ρ above, placing Py for M and putting $m = \left(\frac{a}{a_1}\,\frac{P}{E\,I}\right)$, there results,

$$\frac{1}{\rho} = \frac{d\,\theta}{d\,s} = \left(\frac{a}{a_1}\,\frac{P}{E\,I}\right) y = m\,y \qquad (1)$$

m is here constant for a constant P. Differentiating (1) as to s,

$$\frac{d^2\theta}{ds^2} = m\,\frac{d\,y}{d\,s} = -\,m \sin\theta \qquad (2)$$

Multiplying by $2\,d\,\theta$ and integrating,

$$\left(\frac{d\,\theta}{d\,s}\right)^2 = 2\,m\cos\theta + C.$$

At A'' where $\theta = \theta_0$ and $\frac{d\,\theta}{d\,s} = 0$ from (1) since $y = 0$, $C = -\,2\,m\cos\theta_0$.

$$\therefore \left(\frac{d\,\theta}{d\,s}\right)^2 = 2\,m\,(\cos\theta - \cos\theta_0) \qquad (3)$$

or

$$\frac{d\,\theta}{\sqrt{2\,(\cos\theta - \cos\theta_0)}} = \sqrt{m}\,.\,d\,s \qquad (4)$$

* *Engineering News*, May 6th, 1897, Appendix A (3a).

If θ becomes zero i times between B and A'', the integral of the left member between B and A'' is $i\pi\left(1+\frac{\theta_o^2}{16}\right)$ on neglecting the fourth and higher powers of θ_0 (see Appendix). The integral of the right member between the same limits is $\sqrt{m}\,.\,a_1$; hence to this approximation,

$$i\pi\left(1+\frac{\theta_o^2}{16}\right)=a_1\sqrt{m}\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Equations (1) and (3) applied to points like D, for which $\theta=0$, $y=f$, give $\frac{d\theta}{ds}=mf$; $\left(\frac{d\theta}{ds}\right)^2=2m(1-\cos\theta_o)$; or, since in consequence of θ_o being very small, $2(1-\cos\theta_o)=2(1-1+\frac{1}{2}\theta_o^2)=\theta_o^2$, on neglecting as before the fourth and higher powers of θ_o, the two previous equations give,

$$m^2f^2=m\,\theta_o^2\therefore\theta_o^2=mf^2\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

θ_o can now be eliminated by substituting this value in (5), giving,

$$i\pi\left(1+\frac{mf^2}{16}\right)=a_1\sqrt{m}.$$

On substituting the value of $m=\frac{a}{a_1}\frac{P}{EI}$,

$$f^2=16\frac{a_1}{a}\left[\frac{\sqrt{aa_1}}{i\pi}\sqrt{\frac{EI}{P}}-\frac{EI}{P}\right]\ldots\ldots\ldots\ldots\ldots(7)$$

where $a_1=a\left(1-\frac{P}{EA}\right)=A'CB=A''HDB$ (Fig. 1).

In deriving this formula (7), three approximations have been introduced:

(1) Neglecting the shearing component of P in finding the formula for ρ, (2) and (3) neglecting the fourth and higher powers of θ_o in comparison with θ_o^2 in equations (5) and (6).

As in the practical application of (7) to deriving Euler's formula, f will be supposed very small—as near zero as we please; the errors introduced will not appreciably alter the result.

This formula (7) is given by Bresse* in another form and should be called Bresse's formula. His derivation of (7) offers some objections, which the writer has endeavored to remove in the analysis above. In an article on "Long Columns," in the *Journal of the Franklin Institute*, for July, 1887, the author gave Bresse's original analysis, together

* *Mécanique Appliquée*, première partie, p. 372.

with a discussion and the derivation of other formulas from a different standpoint. It is easy to clear away the apparent obscurity in Bresse's analysis, which furnishes the basis of the preceding discussion, but the method adopted above is so simple and clear that it will doubtless prove more satisfactory. Some very important conclusions follow from equation (7).

The least value of P (call it P_1), at which the preceding theory begins to be applicable, corresponds to $f = 0$ and $i = 1$, corresponding to the simple case of curvature given by Fig. 1:

$$\therefore\ P_1 = \frac{\pi^2 E I}{a\, a_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This is exactly the modified Euler's formula found by Mr. Prichard,* as should be the case; for as f approaches zero, the bent axis tends to coincidence with the straight axis, so that the usual approximation $\frac{1}{\rho} = \frac{d^2y}{dx^2}$, which involves this coincidence in part, should lead to the same formula as that derived from (7) for $f = 0$. For i repetitions of the curvature shown by Fig. 1, $P = \frac{i^2 \pi^2 E I}{a\, a_1}$.

As $a_1 = a\left(1 - \frac{P_1}{E A}\right)$ is very nearly equal to a; on putting a for a_1 in (8), the usual Euler's formula is found. Formula (7) shows that P_1 is the load at which bending just begins, for $i = 1$, or the case of curvature shown by Fig. 1. This very important fact is not brought out by the usual analysis. The superiority in the Bresse analysis over the common one is thus plain, and this gain in interpretation of results is further shown by aid of (7), in proving that a very small increase in P over that given by (8) will cause a sensible deflection and rupture, so that Euler's formula is thus demonstrated to give a value of the load at which not only bending just begins, but also, if a very small proportionate increase in the load is made, rupture will occur; so that Euler's formula is practically a formula for rupture.

As a numerical example take a column composed of two 5-in. channels. $A = 3.9$, $I = 14.8$, $a = 325$ (the inch being the unit), and $E =$ 29 000 000 lbs. per square inch.

From (8), placing a for a_1, $P_1 = 40\,105$ lbs.

Suppose an increase of load of only 5 lbs. $\therefore\ P = 40\,110$ lbs. in (7), where again place $a_1 = a$ approximately, and it is found that

* *Engineering News* for May 6th, 1897, p. 277.

$f = 3.44$ ins. The increase of a few pounds more would lead to rupture, so that (8) gives practically the load corresponding to rupture.

This can be shown more generally in a way suggested, in part, by Mr. Prichard, who kindly gave the author valuable suggestions in criticising his paper before publication.

Call P_2 a value of P slightly greater than P_1, as given by (8), and $a_2 = \text{arc } A'' H D B = A' C B$ (Fig. 1) corresponding to P_2. From (7), replacing a_1 by a_2 and P by P_2 and substituting,

$$a_2 = a_1 - a \frac{P_2 - P_1}{E A} = a_1 \left(1 - \frac{P_2 - P_1}{E A} \right) \text{ very nearly.}$$

$$\therefore a_2 = \frac{\pi^2 E I}{a P_1} \left(1 - \frac{P_2 - P_1}{E A} \right) \text{ by aid of} \ldots\ldots\ldots\ldots (8)$$

$$f^2 = 16 \frac{a_2}{a} \left[\sqrt{1 - \frac{P_2 - P_1}{E A}} \; \frac{E I}{\sqrt{P_1} \sqrt{P_2}} - \frac{E I}{P_2} \right] \ldots\ldots (9)$$

where $\frac{a_2}{a} = 1 - \frac{P_2}{E A} = 1$ nearly.

As an example, using the same cross-section of column as above, $A = 3.9$, $I = 14.8$, $E = 29\,000\,000$, $P_2 = 40\,004$, $P_1 = 40\,000$,* so that $P_2 - P_1 = 4$ lbs. only; from (9),

$$f = 3 \text{ ins. nearly.}$$

By a well-known formula, the total maximum stress on the concave side (at D, Fig 1), due both to the uniform compression and that caused by flexure, is, $\frac{P_2}{A} + \frac{P_2 f V}{I} = 30\,560$ lbs., so that the limit of elasticity has been slightly exceeded.

An increase of load of only 4 lbs. thus causes the maximum stress per square inch to change from $P_1 \div A = 10\,260$ lbs. to $30\,560$ lbs. It is plain that a few pounds more, say about 10 lbs., added to $P_1 = 40\,000$, would cause rupture.

This shows that Euler's formula gives, not only the load at which bending just begins, but practically the load at which rupture occurs.

The increase in f is so rapid for a very small addition to P_1 that rupture may be said to ensue for P_1 as found from (8), or the ordinary Euler's formula where a_1 is replaced by a.

The two conclusions, first, that Euler's formula gives the load at which bending just begins, and, second, that a very small increase to

* a, the corresponding length of column, can be found from (8) if desired. It is not needed in what follows.

this load insures failure of the column, have been often assumed without proof. They are here rigorously proved.

For the case of the column "fixed at both ends," the line $G\ B\ D\ A''\ H$ (Fig. 3), will represent the axis, the part $B\ D\ A''$ corresponding to the similarly marked part in Fig. 1, and the portions $A''\ H$, $G\ B$, with vertical tangents at G and H, being identical with $D\ A''$ or $B\ D$ inverted. Arc $E\ H\ D\ G\ F$ is a portion of the axis for pivoted ends where $i = 3$ in equation (7); hence, in Fig. 3, P acts along the chord $A''\ B$, giving bending moments at H and G sufficient to insure vertical tangents there.

Regarding, as before, $a_1 = B\ D\ A''$ and $a =$ length of same before compression, then for column $G\ D\ H$ "fixed at ends,"

$2a$ = length of unstrained column $= l$,

$2a_1 =$ " strained " $= l_1$, and

$f' = C'\ D = 2f$.

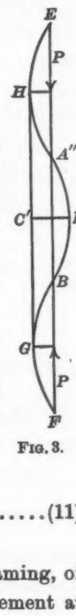

FIG. 3.

In (7) on putting $\frac{1}{2} f'$ for f, $\frac{1}{2}\ l$ for a, and $\frac{1}{2}\ l_1$ for a_1, an expression for f' can be found. The corresponding modified Euler's formula can be found from this directly by making $f' = 0$, or more simply from (8),

$$P_1 = \frac{4\ \pi^2 E I}{2\ a \,.\, 2\ a_1} = \frac{4\ \pi^2 E I}{l\ l_1} \qquad \text{(10)}$$

For the column fixed at one end, pivoted at the other, several cases present themselves. If the upper end is entirely free to move laterally, then $D\ A''$ (Fig. 1) will represent the axis for no repetitions of the simplest case of curvature.

Therefore, from (8),

$$P_1 = \frac{\pi^2 E I}{4 \frac{a}{2} \frac{a_1}{2}} = \frac{\pi^2 E I}{4\ l\ l_1} \qquad \text{(11)}$$

where $l =$ length of unstrained column. If, however, the framing, of which the post constitutes a part, admits little lateral movement at the top of the post, the part of the axis $G\ B\ D\ A''$, Fig. 3 (fixed at G and free at A''), may more closely approximate to the truth. From (8),

$$P_1 = \frac{9}{4}\ \frac{\pi^2 E I}{\frac{3}{2} a \frac{3}{2} a_1} = \frac{9}{4}\ \frac{\pi^2 E I}{l\ l_1} \qquad \text{(12)}$$

where $l =$ unstrained length of axis;

$l_1 =$ strained " " "

In case A'' is compelled to remain in the same vertical line with G, an investigation, not given here, will show that $\frac{9}{4}$ in the last formula will be replaced by 2.05, which differs from it but slightly.

In all of these formulas $l_1 = l\left(1 - \frac{P_1}{E\,A}\right)$. This value can be substituted and the resulting quadratic solved for P, but there is no practical gain in this more exact method. It is always practically exact, where l_1 is involved simply as a factor, to let $l_1 = l$, in which case the preceding formulas (8), (10), (11) and (12), give the loads at which bending begins and failure speedily follows.

If there happens to be a greater number of repetitions of the simple curvature than that assumed above, then P_1 is greater. Pure theory cannot resolve this question, and experiment has to be appealed to in practice to decide upon the most probable case of curvature for the different end conditions.

If l = length of column, r = radius of gyration of cross-section about an axis projected in H (Fig. 1), and if $l_1 = l$ approximately, the preceding formulas can be expressed by

$$P = n\,\frac{\pi^2\,E\,I}{l^2} = n\,\frac{\pi^2\,E\,A\,r^2}{l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

or,

$$\frac{P}{A} = (n\,\pi^2\,E) \div \left(\frac{l}{r}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

where the theoretical values of n generally adopted are, for both ends pivoted, $n = 1$; one end fixed, the other pivoted, $n = \frac{9}{4}$; both ends fixed, $n = 4$, as given above.

Calling S_e the elastic limit of the material, if (14) gives the unit stress on the cross-section $\frac{P}{A} > S_e$, the formula is inapplicable, as it was expressly assumed from the beginning that the limit of elasticity was not to be exceeded.

For a value of $\frac{l}{r}$ in (14) which gives $\frac{P}{A} = S_e$, the column will fail (on a very slight increase of P) by bending, and the same is true for greater values of $\frac{l}{r}$. For less values of $\frac{l}{r}$, $\frac{P}{A} > S_e$.

The limiting value of $\frac{l}{r}$, below which (14) is inapplicable, and

above which it is applicable, is found from (14) by putting $\frac{P}{A} = S_e$,

$$\therefore \text{ limit } \frac{l}{r} = \sqrt{\frac{n \pi^2 E}{S_e}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

At and above this limit, formula (14) gives the average unit stress that practically leads to failure by bending, on supposing lateral forces applied that cause bending and then conceiving the forces removed. Of course, in actual columns, the bending results from crookedness, lack of homogeneity or eccentric application of the load, so that no imaginary force has to be temporarily applied to start the bending. Below the limit (15) two cases appear: one, for a single application of a load causing immediate failure, and, two, for millions of applications of a load leading ultimately to failure.

In the first case, for short, ideal columns, of sufficient length to admit the fractured portions sliding freely off, the average unit stress for failure should correspond to the crushing strength of the material.

Plates XLI and XLII in the paper by T. H. Johnson, M. Am. Soc. C. E., on "The Strength of Columns," * illustrate this view in a general way. Here, the average unit stress causing failure for a single application of a load gradually applied, for $\frac{l}{r}$ between 10 and 30 varied for wrought-iron columns with fixed ends from about 34 000 to 55 000 lbs. per square inch, the average being about 45 000.

Mr. Johnson quotes Mr. Christie as giving the modulus of rupture of wrought-iron beams as 44 800. As the law connecting deformation with stress is not known, when the stress exceeds the elastic limit, no formula for failure of ideal columns below the limit (15) for a single application of the load can be given. At the limit, for the columns just mentioned, $\frac{P}{A} = S_e = 29\,000$ and for $\frac{l}{r} = 20$, $\frac{P}{A} = 45\,000$ say, but between these limits no formula for failure can be given with the present knowledge of the subject.

In the second case cited above, for millions of applications of a load, if it is assumed after Wöhler that any load above the elastic limit, repeated millions of times, will lead to failure, then

$$\frac{P}{A} = S_e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

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* *Transactions* Am. Soc. C. E., Vol. xv, p. 517.

will represent the unit stress leading to failure after millions of repetitions of the load, when $\frac{l}{r}$ is less than the value given by (15).

In this case, if successive values of $\frac{l}{r}$ are laid off as abscissas, and the corresponding values of $\frac{P}{A}$ as ordinates, then the theoretical locus for the failing unit stress for the ideal column will be the straight line (16) for $\frac{l}{r}$ varying from 0 to the value given by (15), after which it coincides with the curve given by Euler's equation.

The fact must not be lost sight of though, that a single application of the critical load given by (14) leads to failure when $\frac{l}{r}$ exceeds the limit (15), while a very great number of applications of the load given by (16) is required for failure for $\frac{l}{r}$ below the limit. The degree of security is thus not the same for the straight and curved portions of the locus. The author is not aware of this distinction having been noted before.

This completes what the author has to say about the ideal column. The actual column is purposely not touched on, as it is first of all necessary to have clear and sound views with reference to the ideal column. This must be the apology for the rather long mathematical discussion which is given. The value of f in (7) is seen to lead to the most important practical conclusions, which may here be recapitulated:

I. Euler's formula is deduced from (7).

II. It follows from (7) that Euler's formula corresponds to incipient bending of the column.

III. By aid of (7) it is shown that a very small increase to the load given by Euler's formula will lead to a considerable bending of the column, and consequent failure from the combined stresses due to the uniform compression and flexure.

IV. Hence, practically, Euler's formula gives the load that causes failure when $\frac{l}{r}$ exceeds a certain limit given by equation (15).

APPENDIX.

From the formula, x being small,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \ldots$$

there results,

$$2(\cos\theta - \cos\theta_o) = \theta_o^2 - \theta^2 - \tfrac{1}{12}(\theta_o^4 - \theta^4)$$
$$= (\theta_o^2 - \theta^2)\left[1 - \tfrac{1}{12}(\theta_o^2 + \theta^2)\right]$$

on neglecting powers of θ higher than the fourth.

Whence,

$$2\,i\int_0^{\theta_o} \frac{d\,\theta}{\sqrt{2\cos\theta - \cos\theta_o}}$$
$$= 2\,i\int_0^{\theta_o} \left[1 - \tfrac{1}{12}(\theta_o^2 + \theta^2)\right]^{-\frac{1}{2}} \frac{d\,\theta}{\sqrt{\theta_o^2 - \theta^2}}$$
$$= 2\,i\int_0^{\theta_o} (1 + \tfrac{1}{24}\theta_0^2 + \tfrac{1}{24}\theta^2)\,\frac{d\,\theta}{\sqrt{\theta_o^2 - \theta^2}}$$
$$= i\,\pi\left(1 + \frac{\theta_0^2}{16}\right)$$

as given in equation (5).

CORRESPONDENCE.

Mr. Marston.

A. MARSTON, Assoc. M. Am. Soc. C. E.—There has recently been a great deal of discussion relating to the theory of the ideal column, centrally loaded, which has accomplished much in the way of clearing up ideas concerning that subject, and in making generally known to engineers many long-established facts which have heretofore been understood by comparatively few. In this discussion, however, the writer has seen nothing relating to what may be called the theory of the ideal column under eccentric loading, though this theory is very essential to a complete understanding of the subject of the strength of columns. In fact the writer is not aware that any correct analysis of the theory of the ideal column eccentrically loaded has ever been made, though it is quite possible that this may have been done. A search in such literature relating to the subject as is accessible to him has indicated that the analysis usually given is based on the pure assumption that under eccentric loading the column bends into a parabolic curve.*

The writer desires to present herewith an exact analysis of the theory of the ideal column with pivoted ends under eccentric loading. The mathematical details will be found at the close of his discussion, but before giving the analysis the writer wishes to state and discuss some of its results.

First.—The deflection at the center of an ideal column with pivoted ends eccentrically loaded is given by the formula

$$f = d\left[\text{Sec.}\ \left(\frac{1}{2}\frac{l}{r}\sqrt{\frac{W}{A\,E}}\right) - 1\right], \qquad (1)$$

while the ultimate strength of such a column is given by the formula

$$\frac{P}{A} = \frac{S_e}{1 + \dfrac{d\,V}{r^2}\ \text{Sec.}\ \left(\dfrac{1}{2}\dfrac{l}{r}\sqrt{\dfrac{P}{A\,E}}\right)} \qquad (2)$$

In the above formulas:

W = any load on the column, not greater than P,

P = the load on the column which makes the greatest stress in the outer fiber equal to the elastic limit,

A = the cross-section of the column,

d = the eccentricity of loading,

V = the distance from the neutral axis to the outer fiber,

l = the length of the column,

I = the moment of inertia of the column section,

r = the radius of gyration of the column section,

S_e = the elastic limit stress of the material,

* See Professor Cain in the *Journal of the Franklin Institute*, Vol. 124, p. 130; also, Professor J. B. Johnson in *Modern Framed Structures*, p. 144.

Mr. Marston.

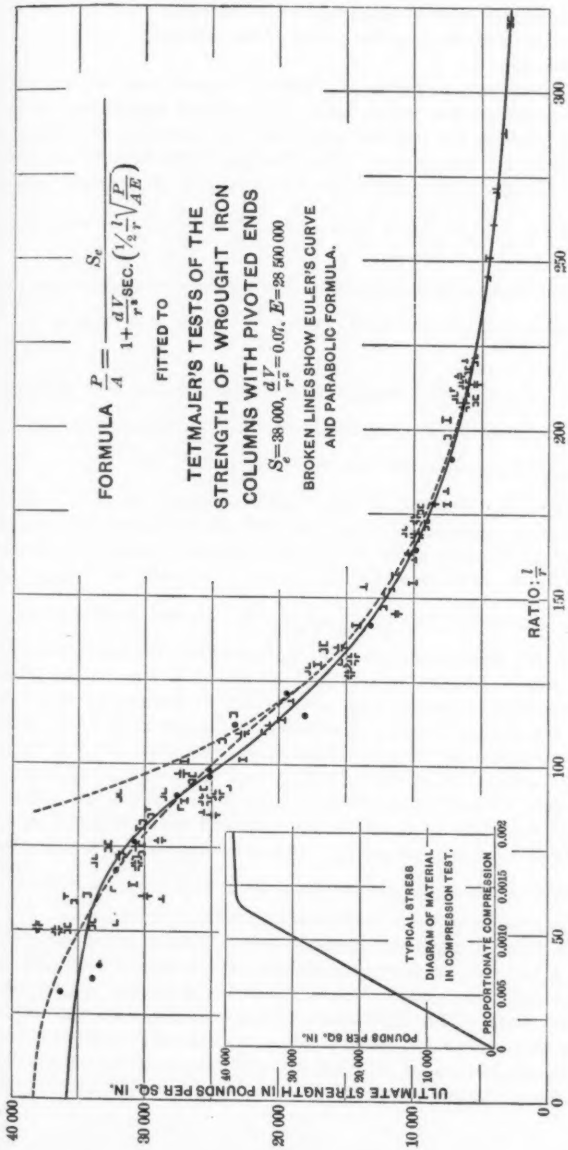

FIG. 4.

Mr. Marston.

E = the modulus of elasticity of the material,

f = the deflection at the center of the column.

See, also, Fig. 6.

Second.—From equation (1), Euler's formula can be derived in a way that shows that if the load on a column equals that given by Euler's formula, the slightest appreciable eccentricity of loading will cause failure of the column. This fact has often been stated without proof. To obtain the derivation, note that f in (1) becomes equal to infinity for $\frac{1}{2}\,\frac{l}{r}\sqrt{\frac{W}{A\,E}} = \frac{\pi}{2}$, *i. e.*, for $\frac{W}{A} = \frac{\pi^2\,E}{\left(\frac{l}{r}\right)^2}$, which is Euler's formula. Evidently for eccentric loading the stress in the outer fiber will reach the elastic limit before $\frac{W}{A}$ becomes as large as $\frac{\pi^2\,E}{\left(\frac{l}{r}\right)^2}$. For the values of $\frac{d\,V}{r^2}$ usually met in practice, however, formula (2) gives results practically identical with Euler's formula for values of $\frac{l}{r}$ greater than about 200 (see Figs. 4 and 5).

Third.—If equation (2) were easy to solve, it would furnish an excellent practical column formula to represent the crippling strength of columns, for pivoted ends centrally loaded, as shown by actual tests. The term in (2) which represents the effect of the eccentricity is $\frac{d\,V}{r^2}\,Sec.\left(\frac{1}{2}\,\frac{l}{r}\sqrt{\frac{P}{A\,E}}\right)$. In the coefficient of this term V may be assumed nearly proportional to r for usual column sections. As to d, it may be said that, no matter how much care is taken, it is impossible to avoid some eccentricity of loading in actual tests. There will always be in actual columns, also, some lack of homogeneity and of straightness which will have much the same effect as eccentricity of loading. It is evident that in a given set of tests in which the same care to avoid the above defects is taken for one test as for another, the value of d, which will represent them, will be larger for large than for small values of r. It seems reasonable in fact, to assume that both d and V are proportional to r, which would make $\frac{d\,V}{r^2}$ an empirical constant, to be determined by actual experiment.

That this assumption is reasonable may be seen by examination of Figs. 4 and 5. In these figures the writer has reproduced from Professor J. B. Johnson's "Materials of Construction," pages 364 and 365, the results of Tetmajer's tests of the strength of wrought iron and of mild steel columns. This is the most extensive series of careful tests, by one person, of columns with pivoted ends, to the results of which the writer has access. The writer has fitted equation (2) to these

Mr. Marston

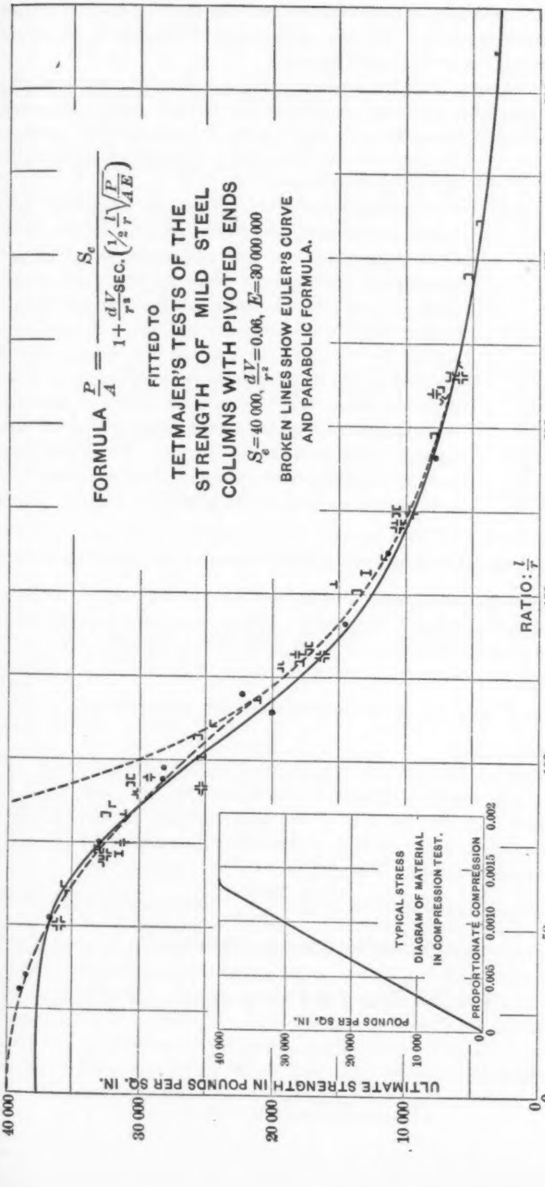

FIG. 5.

Mr. Marston. tests, and has drawn the resulting curves in the figures. He has also, for comparison, drawn Euler's curves and Professor J. B. Johnson's parabolic curves on the same figures.

It will be seen that the curves corresponding to equation (2) correspond very closely to the results of the actual tests. Equation (2) can also readily be made to fit the results of column tests given by T. H. Johnson, M. Am. Soc. C. E., in his paper on "The Strength of Columns."*

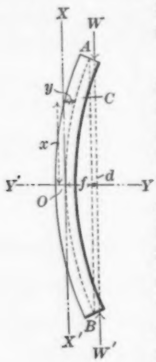

FIG. 6.

Although equation (2) comes nearer to being a perfectly rational practical column formula than any other the writer has seen proposed, and although its agreement with actual tests is all that could be desired, it is nevertheless so difficult to solve that the writer does not advocate its use. It can only be solved by trial and the process is tedious. If it were desired to use it, the proper way would be to evaluate it for all cases likely to occur in practice and plat the resulting curves on a diagram from which all desired values could be read at a glance. The labor required for this would not be excessive, but, even so, it is unnecessary, as well as undesirable, for the following reason:

Fourth.—For all values of $\frac{l}{r}$ likely to occur in practice, the curves of equation (2) very nearly coincide with the curves of Professor J. B. Johnson's parabolic formulas. These are very simple, and the writer advocates their use in preference to all others. The only disagreement of consequence between these formulas is for very small values of $\frac{l}{r}$, and for such conditions there are no tests with pivoted ends.

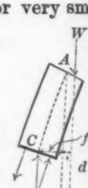

FIG. 7.

Mathematical Derivation of Equations (1) and (2) Above.—For notation, position of axes of co-ordinates, etc., see above, and Fig. 6. In Fig. 7 the portion $A\ C$ of the column is shown as a free body. Let M be the moment of the stress couple at C. $M = W(f + d - y)$. But by the common theory of flexure $M = E\,I\frac{d^2 y}{d x^2}$. Multiplying both values of M by $2\,d\,y$, and prefixing the sign of integration:

$$E\,I\int_0^{d\,y}\frac{2\,d\,y\,d^2 y}{d x^2} = 2\,W\int_0^y (f + d)\,d\,y - W\int_0^y 2\,y\,d\,y \ldots\ldots (a)$$

whence

$$E\,I\,\frac{d\,y^2}{d\,x^2} = 2\,W(f + d)\,y - W\,y^2 \ldots\ldots\ldots\ldots (b)$$

* *Transactions* Am. Soc. C. E., Vol. xv.

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Solving (b) for $d\,x$ and prefixing the sign of integration:

$$\int_0^x d\,x = \sqrt{\frac{E\,I}{W}} \int_0^y \frac{d\,y}{\sqrt{2\,(f+d)\,y - y^2}} \quad \text{(c)}$$

Hence (see Johnson's Calculus, page 9):

$$x = \sqrt{\frac{E\,I}{W}}\ \text{vers.}^{-1} \left(\frac{y}{f+d}\right) \quad \text{(d)}$$

which is the equation of the elastic curve into which the column bends. In the usual analysis of columns under eccentric loading this curve is assumed to be a parabolic arc.

To find f, note that for $x = \frac{1}{2}\,l$, $y = f$. Hence

$$\frac{1}{2}\,l = \sqrt{\frac{E\,I}{W}}\ \text{vers.}^{-1} \left(\frac{f}{f+d}\right) \quad \text{(e)}$$

whence

$$\frac{f}{f+d} = \text{vers.} \left(\frac{1}{2}\ l \sqrt{\frac{W}{E\,I}}\right) = \text{vers.} \left(\frac{1}{2}\,\frac{l}{r} \sqrt{\frac{W}{A\,E}}\right) \quad \text{(f)}$$

$$\frac{f}{f+d} = 1 - \cos. \left(\frac{1}{2}\,\frac{l}{r} \sqrt{\frac{W}{A\,E}}\right) \quad \text{(g)}$$

$$f = d\left[\sec. \left(\frac{1}{2}\,\frac{l}{r} \sqrt{\frac{W}{A\,E}}\right) - 1\right] \quad \text{(1) Q. E. D.}$$

The total stress in the outer fiber on the concave side at the middle of the column will, by the common theory of flexure, be equal to $\frac{W}{A} + \frac{W\,(f+d)\,V}{I}$. When $W = P$, the crippling load on the column, the stress in the outer fiber becomes equal to S_e, the elastic limit stress of the material.

Hence

$$\frac{P}{A} + \frac{P\,(f+d)\,V}{I} = S_e \quad \text{(h)}$$

$$\frac{P}{A} + \frac{P\,d\,V \sec. \left(\frac{1}{2}\,\frac{l}{r} \sqrt{\frac{P}{A\,E}}\right)}{A\,r^2} = S_e \quad \text{(k)}$$

$$\frac{P}{A} = \frac{S_e}{+\,\frac{d\,V}{r^2}\ \sec. \left(\frac{1}{2}\,\frac{l}{r} \sqrt{\frac{P}{A\,E}}\right)} \quad \text{(2) Q. E. D.}$$

Since the above was prepared the writer has accidentally learned that Mr. Carl G. L. Barth, of Scranton, Pa., has made a rigid analysis of columns eccentrically loaded. The writer, having addressed a letter of inquiry to Mr. Barth, learns that this analysis leads to the same formulas as the writer's. Mr. Barth's results have never been published, but in part were presented orally to the Engineers' Club of St. Louis, on May 5th, 1897. It is quite probable that others may have worked out the same formulas.

Mr. Johnson. J. B. JOHNSON, M. Am. Soc. C. E.—The author is to be congratulated on his very clear and concise review and amplification of Euler's formula for ideal long columns. As a purely mathematical discussion it is complete and entirely satisfactory. The danger is always imminent, however, that this, the only purely theoretical column formula which is true in practice should come into practical use. The fact is, it applies only to such great lengths that it can very seldom be employed in any kind of actual designing, and it is therefore of little or no practical value. On the other hand, it is commonly found in works on physics, in engineers' hand-books, and sometimes in treatises on framed structures, without the necessary warnings as to its length limitations. Thus it is ignorantly used for lengths far without its proper field, when it leads to extravagantly high and dangerous working stresses with any ordinary factor of safety. As a working formula, therefore, it has little or no value.

The author also calls attention to the greater degree of security obtained for those lengths to which Euler's formula applies, than would be found for shorter columns, even when an empirical formula is used for these. This subject has also been considered by the writer in the last (sixth) edition of *Modern Framed Structures* (1897), where twice as great a factor of safety is employed for the shorter, or ordinary, lengths as is used for the very great lengths to which Euler's formula applies, his attention having been first called to the rational necessity of this by Mr. Carl G. L. Barth. For the more ready application of these new formulas, diagrams have been prepared and published in this last edition of the work cited above, from which working stresses can be found at once for any end condition, and for all values of $\frac{l}{r}$ up to 300, and for materials having elastic limits from 30 000 to 50 000 lbs. per square inch. Such diagrams could, of course, be prepared for any formula or set of formulas, and, when so prepared, they save a vast amount of work in computing the cross-sections of columns in actual designing.

When the elastic limit of the material is regarded as the ultimate strength of very short columns, a factor of safety of 4 applied to this ultimate is certainly very safe. This factor of safety may be divided into two factors of two each, when one of these may be regarded as a factor to provide for greater loads than those assumed and the other to provide for weaker columns than those assumed. That is to say, provision would then have been made for loads twice as large as those assumed and for structural members half as strong as those assumed. This would be the case for the shorter columns, which are those of ordinary engineering and architectural practice. For the very long columns, the ultimate strength of which is solely a function of their stiffness and not at all a function of their strength, or

of the elastic limit of the material, and for which lengths only the Euler formula applies, it is only necessary to provide against an excess of loading, as the fiber stresses are always very low and far within the elastic limits. Here a factor of safety of something over 2 on the elastic limit is quite sufficient. In the writer's new formulas the factor of safety on the elastic limit gradually reduces from 4, for $\frac{l}{r}$ = zero, to 2 for $\frac{l}{r}$ = infinity. These formulas are as follows: Mr. Johnson.

$$\text{For pivotal ends: } p = \tfrac{1}{2}\left[\frac{f}{2 + \frac{f-p}{10\,E}\left[\frac{l}{r}\right]^2}\right] \ldots\ldots\ldots\ldots (1)$$

$$\text{For pin ends: } p = \tfrac{1}{2}\left[\frac{f}{2 + \frac{f-p}{16\,E}\left[\frac{l}{r}\right]^2}\right] \ldots\ldots\ldots\ldots (2)$$

$$\text{For flat ends: } p = \tfrac{1}{2}\left[\frac{f}{2 + \frac{f-p}{25\,E}\left[\frac{l}{r}\right]^2}\right] \ldots\ldots\ldots\ldots (3)$$

Where p = working stress in lbs. per square inch,
f = elastic limit (commercial) of the material,
E = modulus of elasticity,
l = length of column,
r = least radius of gyration of column.

These formulas come directly from the writer's "rational" formula,* which are applicable to all lengths, and to eccentric loads. It is only by assuming the load to act eccentrically that a rational formula can be found for the ordinary lengths, or for those shorter than the limiting length to which the Euler formula applies.

In the above formulas it will be noted that the factor 2 inside the parentheses has its full effect when $\frac{l}{r} = 0$, and becomes of less and less effect as $\frac{l}{r}$ increases, and is relatively zero, for $\frac{l}{r}$ = infinity. Hence the factor of safety here is 4 for very short and nearly reduces to 2 for very long columns.

Henry S. Prichard, M. Am. Soc. C. E.—The writer has been much interested in the author's analysis, and has carefully examined and verified each step by which his equation (7) is obtained. Mr. Prichard.

The author's analysis and that of Bresse, to which he refers, differ from any others, of which the writer has knowledge, in starting with the correct general expression for the radius of curvature, instead of the close approximation which ordinarily serves as the basis of investigations involving deflection.

* Eq. (1) p. 144, *Modern Framed Structures.*

Mr. Prichard.

The use of the usual approximate instead of the accurate expression for the radius of curvature greatly simplifies analysis, and it is universally recognized that for application to beams, subject to transverse loading, the loss in accuracy from the use of the approximation is very slight if the deflection is quite small, as compared with the length between supports, as it is in nearly all cases which can occur in practice. Hence all the text and reference books in treating of flexure use the approximate instead of the accurate general expression.

Euler obtained his famous formula for columns by starting with the approximate general expression for curvature and equating it with the expression for curvature in terms of the moment of inertia, modulus of elasticity and bending moment; the bending moment being the product of the load on the column into the deflection.

As it is a matter of common experience that columns begin to sensibly deflect under loads much lighter than those causing failure, and that the deflection increases as the load increases, in a way which indicates a connection between the load and the amount of the deflection, it seems natural to expect that a formula for columns will either involve the deflection directly, or indirectly, by substituting for the deflection its value in terms of the intensity of the extreme fiber stress, modulus of elasticity and properties of the cross-section. Nevertheless, although Euler obtained his formula by assuming deflection, it is independent of both the amount of the deflection and the intensity of the extreme fiber stress, as the deflection drops out in the latter part of the analysis by which the formula is obtained.

This fact has led to considerable misunderstanding and discussion regarding the significance of the formula, some contending that it is simply a formula for incipient bending, and others that it is a failure formula. As the results which it gives, when viewed as failure loads, are absurdly high for short columns, and as they are not a function of the allowed intensity of stress, it has been thought by some analysts that in the derivation of the formula compressive stresses have been neglected, and numerous attempts have been made to correct this supposed deficiency.

To the great majority of engineers the formula has been an enigma. Its correct significance, as given by the author, has been understood by some and published from time to time, but the reasons advanced for their understanding have failed to be entirely convincing.

The author, by using the correct general expression for the radius of curvature, has deduced a formula in which the deflection has not dropped out, except for incipient bending, and, by applying it to a fairly typical example and obtaining numerical results, has made it very evident that Euler's formula, with a very slight modification, gives the load at which a column, under ideal conditions, begins to bend if dis-

turbed, and that under practically the same load the column will continue bending until it fails. Mr. Prichard.

It thus furnishes a key to the interpretation of Euler's formula, and it is for this reason, rather than for the magnitude of the gain in accuracy over analyses which start with the approximate expression aforesaid, that his analysis is valuable.

In the example given by the author the numerical error which would result from the use of the approximate expression for curvature is astonishingly small, being less than 5 lbs. in 40 000.

Although the author confines his analysis to the behavior of columns under ideal conditions, and although the conclusions reached can be only even approximately realized in the most careful experiments, and are seemingly contradicted by common experience, nevertheless the cause of accurate knowledge regarding columns as they are used in practice may be greatly furthered by considering this seeming contradiction in connection with the rigid character of his analysis.

Such a consideration must inevitably lead to the conclusion that it is in the differences between the conditions actually existing and the ideal ones assumed, and not in any error or omission in the development of the theory from the assumed conditions, that the explanation is to be found of the seeming contradiction between the theoretical conclusions and the facts as to behavior of columns in practice. When engineers, instead of attempting the inherently impossible task of accounting for the behavior of actual columns under the assumption of purely ideal conditions, modify their assumptions so as to take into account the bending moment which, on account of kinks and bends in the axis, eccentric and transverse application of loads, non-homogeneity of material, etc., exists from the start, to a greater or less extent in every column, they will arrive at a correct understanding of the matter, and will find that the behavior of columns, which from a superficial consideration seems erratic and inexplicable, furnishes one of the strongest experimental proofs of the substantial accuracy of the theory of flexure.

Wm. Cain, M. Am. Soc. C. E.—The aim of this discussion of "The Ideal Column" has been clearly appreciated and stated by Mr. Prichard, and it is to be hoped that it will remove many difficulties that some have found in the usual treatment of the subject; and it may be emphasized here that "the uniform compression" has been included and, further, it has been found that a very few pounds over the load given by Euler's formula will not simply lead to a little harmless bending for very long columns [where $\frac{l}{r}$ is greater than the limit given by (15)], but will cause crippling by direct compression on the concave side, precisely as in the case of shorter columns to which Euler's formula does not apply. Mr. Cain.

Mr. Cain. Professor Johnson has sounded a note of caution as to the improper use of this formula by some authors, for cases to which it is inapplicable, and states that it applies only to such great lengths that it can rarely be used as a working formula, all of which is very true.

The author has great respect for the formulas of Professor Johnson, but an objection can be urged to giving different safety factors, varying from 4 to 2 as $\frac{l}{r}$ varies from 0 to α; for suppose the very long column to be designed with a factor of safety 2, and the short column with the factor 4, and that in the course of time the loads have doubled (such cases have occurred), then the very long column has attained its crippling load, but the short column still has a factor of safety of 2 and perhaps more, if the original factor referred to the elastic limit of the material for $\frac{l}{r} = 0$ in place of the so-called crushing strength for a single application of the load. This point is clearly brought out in the original paper.

The author is gratified that Mr. Marston has given in his discussion the theory for the "Ideal Column" eccentrically loaded. It completes the subject. It has been shown that using the correct value for the radius of curvature, essentially the same formula (Euler's) is deduced as when the usual approximation, $1 \div \frac{d^2 y}{d x^2}$, is used. Mr. Marston uses the latter value; his results are essentially exact and are deduced in the simplest manner.

The resulting formula (1) is to be found in Bresse* and elsewhere, but no one has hitherto constructed the diagrams (Figs. 4 and 5), which are very interesting and instructive, as only one approximation has been introduced, namely, placing $\frac{d\ V}{r^2}$ equal to a constant. The author has discussed the objection that might be urged to this,† as it leads to different values of d in terms of the depth of column, for different shapes, the depth being taken in the direction of the supposed bending.

Thus, for $\frac{d\ V}{r^2} = 0.2$, on substituting known values of V and r, it is found that for rectangular sections, $d = \frac{1}{30}$ depth; for solid cylindrical columns, $d = \frac{1}{40}$ diameter; for thin, hollow cylinders, $d = \frac{1}{20}$ diameter; for thin, hollow square columns, $d = \frac{1}{15}$ depth, and for two channels connected by latticing which give way in the direction of the

* *Mécanique Appliquée*, première partie, p. 384.

† *Journal of the Franklin Institute*, July, 1887.

latticing, $d = \frac{1}{10}$ depth of latticing, approximately. If $\frac{d\,V}{r^2} = 0.07$, the results are 0.35 of the above. Strictly, a separate formula for each shape would alone meet the objection, but when it is remembered that, for the actual column, crookedness and lack of homogeneity have to be considered, as well as eccentricity of the load, and that the formula can only regard all of these abnormalities under the one head of eccentricity, which is, moreover, taken as very small, the objection loses much of its weight, as evidently only a rough average can be obtained in any case. The line of force likewise which has been assumed to act parallel to the axis can just as well be taken as inclined to it or even crossing it. Considering all of these abnormalities, it is really remarkable that the formula fits the experiments so well. Mr. Cain.

An attempt can be made to simplify Mr. Marston's formula (2) by placing

$$\theta = \frac{1}{2}\frac{l}{r}\sqrt{\frac{W}{A\,E}}$$

and developing *sec* θ into a series. Thus changing P to W and S_e to $S =$ total unit fiber stress on concave side of column at mid-length, (2) becomes

$$\frac{W}{A} = \frac{S}{1 + \frac{d\,V}{r^2}\left(1 + \frac{\theta^2}{2} + \frac{5}{24}\theta^4 + \ldots..\right)} \ldots\ldots\ldots\ldots (17)$$

A trial was made to see if $\frac{5}{24}\,\theta^4$ and following terms could be omitted without materially impairing accuracy when $\frac{l}{r}$ was less than, say, 150, but the result was disappointing. Thus, neglecting θ^4, etc., and solving the resulting quadratic [see (18) below] for $\frac{W}{A}$, it is found that Mr. Marston's (2), on replacing P by W and S_e by S, agrees with the approximate value very closely for small values of S and $\frac{l}{r}$. Thus for $\frac{d\,V}{r^2} = 0.07$, $\frac{l}{r} = 50$, $S = 7\,000$; $\frac{l}{r} = 100$, $S = 5\,000$; $\frac{l}{r} = 50$, $S = 10\,000$, the results are practically the same; but for $\frac{l}{r} = 100$, $S = 10\,000$ lbs. per square inch, the approximate method gave about 14% excess over the exact formula, so that it was rejected, even for the working values of S assumed; all the more as S approaches S_e in magnitude.

One matter of interest was evolved from (17), however, on neglecting θ^4 and replacing θ^2 by its value above, giving

$$\frac{W}{A} = \frac{S}{1 + \frac{d\,V}{r^2} + \frac{1}{8\,E}\,\frac{W\,d\,V}{A\,r^2}\,\frac{l^2}{r^2}} \ldots\ldots\ldots\ldots\ldots (18)$$

Mr. Cain When f, the deflection at the center due to flexure only, is very small compared with d (Fig. 6), which obtains for short columns, the unit stress on the concave side due to flexure only is

$$p_2 = \frac{W(f+d)\,V}{I} = \frac{W\,d\,V}{A\,r^2} \text{ nearly.}$$

On substituting this value in (18), there is found the identical formula [see (19) below] given by the author* when adapted to the case of the column pivoted at the ends.

This formula was very simply deduced by assuming the neutral axis to be a parabolic curve, and it is seen to be correct when $\frac{l}{r}$ is small. The parabola was selected for the eccentric loading because the curvature is not zero at the ends of the column as is the case with the sinusoid or the elastic curve for a simple beam transversely loaded. The factor 8 in (18) is thus seen to be a more correct value than the factor 10, sometimes given.

The formula for a column of length l pivoted at the ends, is derived as follows:

For an assumed parabolic neutral axis, the radius of curvature at the center $\rho = \frac{l^2}{8f}$; also $\rho = \frac{E\,V}{p^2}$ [see Merriman's *Mechanics of Materials*, p. 70]. Equating and substituting the derived value of f in

$$S = \frac{W}{A} + p_2 = \frac{W}{A}\left[1 + \frac{(f+d)\,V}{r^2}\right],$$

where S is the total fiber unit stress on the concave side of the column at mid-length, there obtains,

$$\frac{W}{A} = \frac{S}{1 + \frac{d\,V}{r^2} + \frac{p_2\,l^2}{8\,E\,r^2}} \quad \text{.............................} (19)$$

Since $p_2 = \left(S - \frac{W}{A}\right)$, values of $\frac{W}{A}$ can be found, either by trial or by solving the resulting quadratic in $\frac{W}{A}$.

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* *Journal of the Franklin Institute*, July, 1887.